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## Analytic Structure of Amplitudes in Gauge Theories with Confinement <sup>1</sup>

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### Abstract

For gauge theories with confinement, the analytic structure of amplitudes is explored. It is shown that the analytic properties of *physical* amplitudes are the same as those obtained on the basis of an effective theory involving only the composite, physical fields. The corresponding proofs of dispersion relations remain valid. Anomalous thresholds are considered. They are related to the composite structure of particles. It is shown, that there are no such thresholds in physical amplitudes which are associated with confined constituents, like quarks and gluons in QCD. *Unphysical* amplitudes are considered briefly, using propagator functions as an example. For general, covariant, linear gauges, it is shown that these functions must have singularities at finite, real points, which may be associated with confined states.

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## I. INTRODUCTION

The analytic structure of amplitudes is of considerable importance in all quantum field theories, from a physical as well as a conceptional point of view. It has been studied extensively over many years, but mainly for field theories with a state space of definite metric, and in situations where the interpolating Heisenberg fields are closely related to the observable excitations of the theory. Within a relativistic framework of this type, the commutativity or anti-commutativity of the Heisenberg fields, at space-like separations, gives rise to tubes (wedges) of holomorphy for retarded and advanced amplitudes, which are Fourier transforms of tempered distributions. Lower bounds for the spectrum of eigenstates of the energy momentum operator provide real domains where these amplitudes coincide at least in the sense of distributions. One can then use the Edge of the Wedge theorem [1] to show that there exists an analytic function which is holomorphic in the union of the wedges and a complex neighborhood of the common real domain, and which coincides with the advanced and retarded amplitudes where they are defined. Then the envelope of holomorphy [2, 1] of this initial region of analyticity gives the largest domain of holomorphy obtainable from the general and rather limited input. Further extensions require more exhaustive use of unitarity [3], which is often rather difficult. Although the theory of functions of several complex variables is the natural framework for the discussion of the analytic structure of amplitudes, for special cases, like those involving one complex four-vector, more conventional methods, like differential equations and distribution theory, can be used in order to obtain the region of holomorphy [4, 5]. Many more technical details are involved in the derivation of analytic properties and dispersion relations [6, 7], [5, 1, 8, 9], but the

envelopes mentioned above are at the center of the problem.

It is the purpose of this paper to discuss essential aspects of the analytic structure of amplitudes for gauge theories, which require indefinite metric in a covariant formulation, and for which the physical spectrum is not directly related to the original Heisenberg fields. Rather, these fields correspond to unphysical, confined excitations in the state space of indefinite metric. We will be mainly concerned with physical amplitudes, corresponding to hadronic amplitudes for QCD. We will often use the language of QCD. As examples of unphysical Green's functions, we consider the structure functions of the gluon and quark propagators. In all covariant, linear gauges, we show that these generally cannot be entire functions, but must have singularities which can be related to unphysical states. A preliminary account of some of our results may be found in [10].

In the framework of hadronic field theory with positive definite metric, the derivation of analytic properties, and of corresponding dispersion relations, is on a quite rigorous basis, and uses only very general aspects of the theory. For gauge theories with confinement however, the derivation of dispersion relations requires several assumptions, which may not have been proven rigorously in the non-perturbative framework required in the presence of confinement. We will discuss these assumptions in the following sections. They mainly concern the definition of confinement on the basis of the BRST-algebra, and the construction of composite hadron fields as products of unphysical Heisenberg fields.

In order to provide for the input for proofs of dispersion relations and other analyticity properties of physical amplitudes, we discuss in the following paragraphs certain results for non-perturbative gauge theories. For some of these results, we can refer to the literature for detailed proofs, but we have

to explore their relevance for the derivation of analytic properties and for the identification of singularities. It is not our aim, to provide reviews of several aspects of non-perturbative gauge theories. What we need is to show that there is a mathematically well defined formulation of confinement, which we use in order to derive the spectral conditions, and a construction of local, composite fields for hadrons, from which we obtain the initial domain of holomorphy. We also have to exhibit the assumptions made within this framework.

For the purpose of writing Fourier representations for hadronic amplitudes, we must consider the construction of local, BRST-invariant, composite Heisenberg fields corresponding to hadrons. This problem requires a discussion of operator products [11, 12] of elementary, confined fields in a non-perturbative framework.

Since we need a manifestly covariant formulation of the theory, we consider linear, covariant gauges within the framework of the BRST-algebra [13]. Assuming the existence and the completeness [14] of a nilpotent BRST-operator  $Q$  in the state space  $\mathcal{V}$  of indefinite metric, we define an invariant physical state space  $\mathcal{H}$  as a cohomology of  $Q$ . As a consequence of completeness, which implies that all neutral (zero norm) states satisfying  $Q\Psi = 0$  are of the form  $\Psi = Q\Phi, \Phi \in \mathcal{V}$ , the space  $\mathcal{H}$  has (positive) definite metric [15, 16].

We assume that there are hadronic states in the theory, and take confinement to mean that, in a collision of hadrons, only hadrons are produced. Within the BRST-formalism, this implies that only hadron states appear as physical states in  $\mathcal{H}$ . At least at zero temperature, transverse gluons and quarks are confined for dynamical reasons, forming non-singlet representations of the BRST-algebra in combination with other unphysical fields. In

the decomposition of an inner product of physical states, there appear then only hadron states. The same is true for the decomposition of physical matrix elements of products of BRST-invariant operator fields, because these operators map physical states into other physical states. In this way we find that the fundamental *spectral conditions* for hadronic amplitudes are the same as in the effective hadronic field theory. The absorptive thresholds are due only to hadronic states.

There is, however, another category of singularities, which is related to the structure of particles as a composite system of other particles. These are the so-called anomalous thresholds or structure singularities. They were encountered in the process of constructing examples for the limitations of proofs for dispersion representations [18, 19]. These limitations are related to anomalous thresholds corresponding to the structure of a given hadron as a composite system of non-existing particles, which are not excluded by simple spectral conditions. But physical anomalous thresholds [20, 19, 21] are very common in hadronic amplitudes: the deuteron as a np-system,  $\Lambda$  and  $\Sigma$  hyperons as KN systems, etc. In theories like QCD, the important question is, whether there are structure singularities of hadronic amplitudes which are related to the quark-gluon structure of hadrons. We show that this is *not* the case. Independent of perturbation theory, we describe how anomalous thresholds are due to poles or absorptive thresholds in crossed channels of other hadronic amplitudes, which are related to the one under consideration by analytic continuation into an appropriate lower Riemann sheet [21]. Since, as explained above, we have no absorptive singularities in hadronic amplitudes which are associated with quarks and gluons, we also have no corresponding anomalous thresholds.

For form factors of hadrons, which may be considered as loosely bound

systems of heavy quarks, there are interesting consequences stemming from the absence of anomalous thresholds associated with the quark-gluon structure. In a constituent picture, these hadrons can be described by a Schrödinger wave function with a long range due to the small binding energy. But in QCD, in contrast to the situation for the deuteron, for example, there are no anomalous thresholds associated with the spread-out quark structure. However, there is no problem in obtaining a large mean square radius with an appropriate form of the discontinuities associated with hadronic branch lines. In addition, there may be *hadronic* anomalous thresholds which are relevant.

Finally, we consider the analytic properties and the singularity structure of unphysical (colored) amplitudes. It is sufficient to discuss two-point functions as examples. The structure functions of quark and transverse gluon propagators are analytic in the  $k^2$ -plane, with cuts along the positive real axis. This is a direct consequence of Lorentz covariance and spectral conditions. In previous papers [22], we have derived the asymptotic behavior of these functions for  $k^2 \rightarrow \infty$  in all directions of the complex  $k^2$ -plane, and for general, linear, covariant gauges [23]. With asymptotic freedom, they vanish in these limits. Hence the structure functions cannot be non-trivial, entire functions. They must have singularities on the positive, real  $k^2$ -axis, which should be associated with appropriate unphysical (colored) states in the general state space  $\mathcal{V}$  with indefinite metric. These states are not elements of the physical state space, but form non-singlet representations of the BRST-algebra.

## II. CONFINEMENT

In this section, we briefly define the general framework for the later discussion of amplitudes and their analytic structure.

We consider quantum chromodynamics and similar theories. Since it is essential to have a manifestly Lorentz-covariant formulation, we use covariant gauges as defined by a gauge fixing term

$$\mathcal{L}_{GF} = B \cdot (\partial_\mu A^\mu) + \frac{\alpha}{2} B \cdot B , \quad (1)$$

where  $B$  is the Nakanishi-Lautrup auxiliary field, and  $\alpha$  is a real parameter. The theory is defined in a vector space  $\mathcal{V}$  with indefinite metric. We assume that the constrained system is quantized in accordance with the BRST-algebra [13] :

$$Q^2 = 0, \quad i[Q_c, Q] = Q , \quad (2)$$

where  $Q$  is the BRST-operator, and  $Q_c$  the ghost-number operator. On the basis of this algebra, we define the subspaces

$$\ker Q = \{ \Psi : Q\Psi = 0, \Psi \in \mathcal{V} \} , \quad (3)$$

$$\text{im} Q = \{ \Psi : \Psi = Q\Phi, \Phi \in \mathcal{V} \} , \quad (4)$$

where  $\text{im} Q \perp \ker Q$ , with respect to the indefinite inner product  $(\Psi, \Phi)$ . We can write

$$\ker Q = \mathcal{V}_p \oplus \text{im} Q , \quad (5)$$

and define the BRST-cohomology

$$\mathcal{H} = \frac{\ker Q}{\text{im} Q} \quad (6)$$

as a covariant space of equivalence classes, which is isomorphic to  $\mathcal{V}_p$  [15]. We are interested in zero ghost number, and hence ignore the grading due to the ghost number operator  $Q_c$ . In order to use  $\mathcal{H}$  as a physical state space, it must have definite metric, which we can choose to be positive. This is not assured, a priori, but requires the assumption of “completeness” of the BRST-operator  $Q$ , which implies that all neutral (zero norm) states in  $\ker Q$  are contained in  $\text{im} Q$  [14, 15]. Then  $\mathcal{V}_p$  and hence  $\mathcal{H}$  must be definite, because every space with indefinite metric contains neutral states. With completeness, we have  $\text{im} Q = (\ker Q)^\perp$ , and hence  $\text{im} Q$  is the isotropic part of  $\ker Q$ . It is not enough for the definiteness of  $\mathcal{H}$  to have ghost number zero, since ‘singlet pair’ representations, containing equivalent numbers of ghosts and anti-ghosts, must also be eliminated. In view of the inner product for eigenstates of  $iQ_c$  :

$$(\Psi_{N_c}, \Psi_{N'_c}) = \delta_{-N_c, N'_c} , \quad (7)$$

they would give rise to an indefinite metric in  $\mathcal{V}_p$ , and hence to neutral states.

There are arguments for the absence of singlet pairs in the dense subspace generated by the field operators. But we are dealing with a space of indefinite metric, so that the extension to the full space  $\mathcal{V}$  is delicate [25, 24, 26]. Completeness can be proven explicitly in certain string theories, however these are more simple structures than four-dimensional gauge theories. In any case, without completeness, we cannot get a physical subspace with definite norm, and a consistent formulation of the theory would seem to be impossible.

Given completeness, physical states  $\Psi_p$  are BRST-singlets with  $Q\Psi_p = 0$ , positive norm and ghost number zero. Unphysical states form quartet



representations of the BRST-algebra [15]:

$$\Psi_{N_c} \quad \text{with} \quad Q\Psi_{N_c} \neq 0 , \quad (8)$$

$$\Xi_{N_c+1} = Q\Psi_{N_c} , \quad (9)$$

$$\Psi_{-N_c-1} \quad \text{with} \quad Q\Psi_{-N_c-1} \neq 0 \quad \text{and} \quad (\Xi_{N_c+1}, \Psi_{-N_c-1}) \neq 0 , \quad (10)$$

$$\Xi_{-N_c} = Q\Psi_{-N_c-1} \quad \text{and} \quad (\Xi_{-N_c}, \Psi_{N_c}) \neq 0 . \quad (11)$$

The states  $\Psi_{-N_c-1}$  and  $\Xi_{-N_c}$  are implied by the non-degeneracy of  $\mathcal{V}$ , and the inner product in Eq.(7).

In weak coupling perturbative theory, the state space  $\mathcal{H}$  consists of quarks and transverse gluons. Ghosts, longitudinal- and timelike gluons form quartet representations, and are unphysical. They are confined in a kinematical fashion.

In a general non-Abelian gauge theory like QCD, we can have asymptotic freedom, and we expect that all colored states are confined, provided the number of matter fields is limited. In the language of QCD, this implies that quarks and transvers gluons, at zero temperature, are not elements of the physical state space  $\mathcal{H}$ , which then contains only hadrons as colorless, composite systems [25, 17, 16]. Under these circumstances, only hadrons can be produced in a collision of hadrons. This algebraic notion of confinement should be compatible with more intuitive pictures of quark confinement, and with two-dimensional models. However, for gluons, two-dimensional models are useless, because there are no transvers gluons in two dimensions. If the number of flavors in QCD is limited, we can give arguments that gluons

are not elements of the physical subspace [16, 17]. These arguments are based upon superconvergence relations satisfied by the structure function of the gluon propagator, which provide a connection between short- and long distance properties of the theory [22, 23, 27].

In our discussion of analytic properties of hadronic amplitudes, we take it for granted that confinement is realized in the sense that the physical state space  $\mathcal{H}$  contains only hadronic states. Quarks and gluons are not BRST-singlets. Together with other unphysical states, they form quartet representations of the BRST-algebra and remain unobservable.

### III. LOCAL HADRONIC FIELDS

Having defined the general state space of the gauge theory with confinement, we now turn to the problem of constructing local Heisenberg operators, which can be used as interpolating fields in amplitudes describing reactions between physical particles (hadrons), and in form factors of hadrons. The construction of composite operators, and of operator product expansions, has been discussed extensively in the literature [11, 12]. The relatively new aspects in our case are the state space of indefinite metric, and the fact that the constituents are unobservable. In addition, in view of confinement, we cannot use perturbation theory methods, and consequently some assumptions are needed for the non-perturbative construction of composite fields.

In the following, we discuss the problem with the help of a generic example. We consider the construction of a meson field  $B(x)$  in terms of fundamental fields  $\psi(x)$  and  $\bar{\psi}(x)$ , ignoring all inessential aspects like indices etc.. Hence, our formulae in the following are rather symbolic. The field  $B(x)$

must be local and BRST-invariant, so that  $B(x)\Psi$  is a representative of a physical state, provided  $\Psi$  is one.

Let us first consider the product

$$B(x, \xi) = \psi(x + \xi)\bar{\psi}(x - \xi) . \quad (12)$$

With  $|k, M\rangle$  being a one particle hadron state with  $k^2 = M^2$ , we assume that this state exists as a composite system, so that we have a non-vanishing matrix element

$$\langle 0|B(x, \xi)|k, M\rangle \neq 0 , \quad (13)$$

where  $|0\rangle$  denotes the vacuum state, and where the inner product involved in Eq.(13) is the indefinite product defined in the general state space  $\mathcal{V}$ . We now define a Poincaré covariant, local operator by the weak limit

$$B_F(x) = \lim_{\xi \rightarrow 0} \frac{\psi(x + \xi)\bar{\psi}(x - \xi)}{F(\xi)} . \quad (14)$$

We may consider a space-like approach with  $\xi^2 < 0$ , but this is not essential. The invariant function  $F(\xi)$  is only of relevance as far as its singularity for  $\xi \rightarrow 0$  is concerned. It is the purpose of  $F(\xi)$  to compensate the singularity of the operator product. Writing

$$F(\xi) = (\Psi, B(0, \xi)\Phi) , \quad \Psi, \Phi \in \mathcal{V} , \quad (15)$$

we want to choose these states so that they belong to a class  $\mathcal{K}_{max}$ , for which the matrix element (15) is most singular, assuming that such most singular matrix elements exist [28]. Possible oscillations in the limit (14) may require the choice of an appropriate sequence  $\{\xi_n\}$  in the approach to  $\xi = 0$ . By construction, the operator  $B_F(x)$  is local with respect to the constituent fields  $\psi(x)$  and  $\bar{\psi}(x)$ , and with respect to itself.

In view of the requirement (13), we have

$$\langle 0|B(x, \xi)|k, M\rangle = e^{-ik \cdot x} \langle 0|B(0, \xi)|k, M\rangle = e^{-ik \cdot x} F_k(\xi) , \quad (16)$$

with  $F_k(\xi) \neq 0$ . Then the operator field

$$B(x) = \lim_{\xi \rightarrow 0} \frac{B(x, \xi)}{F_k(\xi)} \quad (17)$$

has a finite matrix element. We may assume that  $F_k(\xi) \in \mathcal{K}_{max}$ , so that  $B(x)$  appears as the leading term in the general operator product expansion of  $B(x, \xi)$ . However, by construction, the field  $B_F(x)$  should be a BRST-invariant operator. Since we are dealing only with matrix elements of  $B_F(x)$  with respect to states in the physical state space  $\mathcal{H}$ , it is sufficient to assume that  $F_k(\xi) \in \mathcal{K}'_{max}$ , where  $\mathcal{K}'_{max}$  refers only to states in  $\mathcal{H}$ . We then use the field  $B(x)$  in Eq.(17) as the Heisenberg field interpolating between the corresponding asymptotic states. We introduce asymptotic fields  $B_{in}(x)$  using the free retarded function  $\Delta_R(x - x', M)$  in the Yang-Feldman representation, and apply the conventional LSZ-reduction formalism [29] in order to obtain representations of physical amplitudes in terms of products of  $B(x)$  fields. An example would be the S-matrix element

$$\begin{aligned} \langle k', M; p', M | S | k, M; p, M \rangle &= \frac{1}{(2\pi)^3} \int \int d^4 x' d^4 x \exp[ik' x' - ikx] \\ &\times K_{x'} K_x \langle p', M | T B(x') \overline{B}(x) | p, M \rangle , \quad (18) \end{aligned}$$

or corresponding expressions in terms of retarded or advanced products. We can also make further reductions as required for the proofs of dispersion representations. The reduction method is used here only within the space  $\mathcal{H}$  with definite metric and in a framework without infrared problems.

In four dimensions, the existence of operator expansions, and of composite operators like  $B(x)$ , can be proven within the framework of renormalized

perturbation theory [28], but not yet in the general theory, as required for our purpose. Hence we have to make the technical assumptions described above. In many lower dimensional theories, operator product expansions are known to exist independent of perturbation theory. They are expected to be a general property of local field theories.

The construction of interpolating, hadronic Heisenberg operators, like  $B(x)$  in our example, is of course not unique. But the different possibilities belong to the same Borchers class [30]. Different fields in a given class, which have the same asymptotic fields, define the same S-matrix. It can be shown that locality is a transitive property: two fields, which commute with a given local field, are local themselves and with respect to each other. We have equivalence classes of local fields. Whatever the construction of a composite operator like  $B(x)$ , the resulting fields all are local with respect to the fundamental fields, and hence belong to the same Borchers class. Although we use Borchers theorem here essentially only in the physical subspace, it can be generalized to spaces with indefinite metric. The proof involves the equivalence of weak local commutativity and CPT-invariance, as well as the Edge of the Wedge Theorem. Introducing appropriate rules for the transformation of ghost field under CPT, we can define an anti-unitary CPT-operator in the state space  $\mathcal{V}$ . Together with the postulates of indefinite metric field theory, we then get equivalence classes of local Heisenberg fields in gauge theories like QCD.

The construction of composite hadron fields, as described above, can be generalized to other products of fundamental fields which form color singlets.

## IV. SPECTRAL CONDITIONS

In the previous section, we have described how we can obtain representations of hadronic amplitudes in QCD in terms of local Heisenberg fields, which are BRST-invariant and interpolate between asymptotic states of non-interaction hadrons. While the local commutativity of the hadron fields implies support properties of Fourier representations which give rise to tubes (wedges) as regions of holomorphy, the spectral conditions define the real domain where retarded and advanced amplitudes coincide, generally in the sense of distributions. Given completeness of the BRST-operator  $Q$ , it is convenient for our further discussion to introduce a self-adjoint involution  $\mathcal{C}$  in  $\mathcal{V}$ , which converts the indefinite inner product into a definite product denoted by

$$(\Psi, \Phi)_{\mathcal{C}} = (\Psi, \mathcal{C}\Phi) , \quad (19)$$

where  $\mathcal{C}^\dagger = \mathcal{C}$  and  $\mathcal{C}^2 = 1$  [15, 14, 16]. With respect to the definite product, we obtain a decomposition of  $\mathcal{V}$  in the form

$$\mathcal{V} = \mathcal{V}_p \oplus imQ \oplus imQ^* , \quad (20)$$

where  $Q^* = \mathcal{C}Q\mathcal{C}$  and  $Q^{*2} = 0$ . With completeness of  $Q$ , the subspace  $\mathcal{V}_p$  has (positive) definite metric, while  $imQ$  and  $imQ^*$  contain conjugate pairs of neutral (zero norm) states, so that for every  $\Psi \in imQ$ , there is a  $\Psi' \in imQ^*$  with  $(\Psi, \Psi') \neq 0$ , while both states are orthogonal to  $\mathcal{V}_p$ . Here and in the following we ignore the grading due to the ghost number operator, since we are mainly interested in  $N_c = 0$ . It is convenient to introduce a matrix

notation, writing a vector  $\Psi \in \mathcal{V}$  in the form

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \quad (21)$$

with components referring to the subspaces  $\mathcal{V}_p$ ,  $imQ$  and  $imQ^*$  of the decomposition (20). Then

$$\mathcal{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad (22)$$

and the inner product is given by

$$(\Psi, \Phi) = (\Psi, \mathcal{C}\Phi)_{\mathcal{C}} = \psi_1^* \phi_1 + \psi_2^* \phi_3 + \psi_3^* \phi_2. \quad (23)$$

The BRST-operator can be written as

$$Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & q \\ 0 & 0 & 0 \end{pmatrix}, \quad (24)$$

with  $q$  being an invertible suboperator [16].  $\Psi \in kerQ$  is characterized by  $\psi_3 = 0$ , and a representative of a physical state  $\Psi \in \mathcal{H}$  by  $\psi_3 = 0$ ,  $\psi_1 \neq 0$ . Hence, for  $\Psi, \Phi \in \mathcal{H}$ , we have  $(\Psi, \Phi) = \psi_1^* \phi_1$  in Eq.(23). Since  $\mathcal{V}_p$  is a non-degenerate subspace, we can introduce a projection operator  $P(\mathcal{V}_p)$  with  $P^2 = P^\dagger = P$ .

For the purpose of spectral condition, we are interested in the decomposition of an inner product with respect to a complete set  $\{\Psi_n\}$  of states in  $\mathcal{V}$ , in particular eigenstates of the energy momentum operator. For  $\Psi, \Phi \in \mathcal{V}$ , we have then

$$(\Psi, \Phi) = \sum_n (\Psi, \Psi_n)(\Psi_n, \Phi). \quad (25)$$

But if we consider only states  $\Psi, \Phi \in \ker Q$ , we obtain  $(\Psi, \Phi) = \psi_1^* \phi_1 = (\Psi, P(\mathcal{V}_p)\Phi)$ , so that we can write

$$\begin{aligned} (\Psi, \Phi) &= \sum_n (\Psi, P(\mathcal{V}_p)\Psi_n)(P(\mathcal{V}_p)\Psi_n, \Phi) \\ &= \sum_n (\Psi, \Psi_{pn})(\Psi_{pn}, \Phi), \end{aligned} \quad (26)$$

with a complete set of states  $\{\Psi_{pn}\}$  in the Hilbert space  $\mathcal{V}_p$ . Writing symbolically  $\Psi_{\mathcal{H}n} = \Psi_{pn} + imQ$ , we have  $(\Psi, \Psi_{\mathcal{H}n}) = (\Psi, \Psi_{pn})$  for  $\Psi \in \mathcal{H}$ , and hence obtain the decomposition

$$(\Psi, \Phi) = \sum_n (\Psi, \Psi_{\mathcal{H}n})(\Psi_{\mathcal{H}n}, \Phi), \quad (27)$$

with  $\Psi, \Phi \in \mathcal{H}$ . The expression (27) is manifestly Lorentz invariant, even though the projection  $P(\mathcal{V}_p)$  by itself is not invariant. In the full state space  $\mathcal{V}$  of indefinite metric, Lorentz transformations are realized by unitary mappings  $U$  with  $U^\dagger = \mathcal{C}U^*\mathcal{C}$ . They are BRST-invariant, and consequently of the form (28) in our matrix representation. It is then easy to see that only  $U_{11}$  appears in the transformation of physical quantities. Transformations  $U$  with  $U_{11} = 1$  are equivalence transformations which do not change physical matrix elements. Unphysical states, written as vectors like in Eq.(21), may well have a component in  $\mathcal{V}_p$ , but we can always find an equivalence transformation which removes this component, because  $\psi_3 \neq 0$  for these states.

As we have seen in the previous section, we can obtain hadronic amplitudes as Fourier Transforms of matrix elements involving only BRST-invariant, local hadronic fields and hadron states. All spectral conditions result from decompositions of these products with respect to intermediate states, which are eigenstates of the energy momentum operator. A BRST-invariant operator  $O$  commutes with  $Q$ , and leaves the subspace  $\ker Q$  in-



variant. In our matrix notation, it is of the form

$$O = \begin{pmatrix} O_{11} & 0 & O_{13} \\ O_{21} & O_{22} & O_{23} \\ 0 & 0 & O_{33} \end{pmatrix}, \quad (28)$$

with  $O_{22}q = qO_{33}$ , where  $q$  is defined in Eq.(24). Since  $O\psi \in \mathcal{H}$  if  $\Psi \in \mathcal{H}$ , we can use Eq.(27) to write decompositions of the form

$$(\Psi, XY\Phi) = \sum_n (\Psi, X\Psi_{\mathcal{H}n})(\Psi_{\mathcal{H}n}, Y\Phi), \quad (29)$$

where  $\Psi, \Phi \in \mathcal{H}$  and  $X, Y$  are BRST-invariant operators (fields). We see again, that only physical states appear in the decomposition.

Eq.(29) is generic for all spectral decompositions used in the derivation of analytic properties of physical amplitudes. It shows that these spectral conditions involve only hadrons, and it guarantees the unitarity of the S-matrix [15] in the physical (hadronic) state space  $\mathcal{H}$ . The described features of hadronic amplitudes are, of course, a direct consequence of our assumption of confinement for transvers gluons and quarks.

With the local hadronic operator and hadronic spectral conditions, we have reached the conclusion, that the derivation of analytic properties, and of dispersion representations for gauge theories with confinement, can proceed along the same lines as in the old hadron field theory. The starting point are Fourier transforms of matrix elements of retarded and advanced products of the BRST-invariant, composite, local hadron fields.

However, one important aspect remains to be discussed: the question of anomalous thresholds or structure singularities, which will be considered in the following section.

## V. ANOMALOUS THRESHOLDS

In the literature, anomalous thresholds are often considered in connection with appropriate Feynman graphs [20, 19, 21]. However, they can be understood, completely independent of perturbation methods, on the basis of analyticity and unitarity [21]. Within the framework of theories with confinement, it is essential to have a nonperturbative approach.

Anomalous thresholds are branch points which appear in a given channel of an amplitude. They are not directly related to the possible intermediate states in this channel, which introduce only “absorptive” singularities. They are rather “structure singularities”, which describe effects due to the possibility that a given particle can be considered as a composite system of other particles. They appear in the physical sheet of the amplitude, in the channel considered, only if a loosely bound composite system is involved. Otherwise, they remain in a secondary Riemann sheet.

In the following we briefly show that anomalous thresholds, in a given channel of an amplitude, are due to ordinary (absorptive) thresholds in crossed channels of other amplitudes, which are related to the one under consideration via unitarity. Since we have seen before that hadronic amplitudes have only absorption thresholds related to hadron states, it follows that the only anomalous thresholds which appear are due to the structure of hadrons as composite systems of hadronic constituents. There are no such thresholds associated with the quark-gluon structure of hadrons, even for loosely bound composite systems with quarks as constituents.

In order to study the emergence of anomalous thresholds on the physical sheet of an hadronic amplitude, we consider a form factor as an example. We ignore all inessential complications and use the structure function  $W(s)$ ,  $s =$

$k^2$  of a deuteron-like particle with variable mass, as indicated in Fig.1. For  $x < 2m_N^2$ , where  $x = (mass)^2 \leq m_D^2$ , the function  $W(s)$  has branch points on the right hand, real  $k^2$ -axis, starting with those due to pion intermediate states at  $s_\pi$ . However, we concentrate on the  $N\bar{N}$ -threshold at  $s = 4m_N^2$ . The discontinuity due to this threshold alone is

$$ImW_{N\bar{N}}(s+i0) = \rho(s+i0)G(s+i0)V^{II}(s+i0) , \quad (30)$$

$$\rho(s) = [(s - 4m_N^2)s^{-1}]^{1/2} . \quad (31)$$

Here  $G(s)$  is the appropriate partial wave projective of the amplitude  $G(s, t)$  pictured in Fig.2a. We consider S-wave projections for simplicity. Furthermore  $V^{II}(s+i0) = V^*(s+i0)$ , where  $V(s)$  is the nucleon form factor, with branch points analogous to those of  $W(s)$ . The continuation of  $V(s)$  into sheet  $II$  of the  $N\bar{N}$ -threshold is given by [21]

$$V^{II}(s) = \frac{V(s)}{1 + 2i\rho(s)F(s)} , \quad (32)$$

where  $F(s)$  is the partial wave projection of the scattering amplitude  $N\bar{N} \rightarrow N\bar{N}$  in the s-channel. With Eqs. (30) and (32), we get for the continuation of  $W(s)$  through the  $N\bar{N}$  cut:

$$W^{II}(s) = W(s) - 2i\rho(s)\frac{G(s)V(s)}{1 + 2i\rho(s)F(s)} . \quad (33)$$

While  $W(s)$  has no left-hand branch lines,  $W^{II}(s)$  does, due to left-hand cuts of  $G(s)$  and  $F(s)$ . For our purpose, the important left-hand cut is the one of  $G(s)$ , which is caused by the pole term at  $t = m_N^2$ , as illustrated in Fig.2b :

$$G(s, t) = \frac{\Gamma^2(x)}{m_N^2 - t} + \dots . \quad (34)$$

The branch point is due to the end point at  $\cos\theta = -1$  in the partial-wave projection, with  $t = t(s, \cos\theta)$ . It is located at  $s = g(x)$ , where

$$g(x) = 4x \left( 1 - \frac{x}{4m_N^2} \right) . \quad (35)$$

For  $0 < x < 4m_N^2$ , we have  $g(x) < 4m_N^2$ , with the maximum at  $g(2m_N^2) = 4m_N^2$ . The branch point in sheet II at  $s = g(x)$  is pictured in Fig.3 for  $x < 2m_N^2$ .

Let us now increase  $x$  to  $x = 2m_N^2$  and above. The position  $g(x)$  of the branch point moves to  $4m_N^2$  at  $x = 2m_N^2$ , and then decreases again. Giving  $x$  an imaginary part, we get

$$g(x + iy) = \left( g(x) + \frac{y^2}{m_N^2} \right) - 2i \frac{y}{m_N^2} (x - 2m_N^2) , \quad (36)$$

and we see that  $g(x + iy)$  encircles the branch point  $s = 4m_N^2$  of  $W(s)$ , moving thereby into the first and “physical” sheet of the Riemann surface of this function. There it becomes an anomalous threshold. The situation is illustrated in Fig.4, where the meson cuts have been omitted. For sufficiently large values of  $x$ , this branch point can move well below the lowest absorption threshold  $s_\pi$ . Writing  $m_D = 2m_N - B$ , we get, for small values of the binding energy  $B$ ,

$$g(m_D^2) = 4m_D^2 \left( 1 - \frac{m_D^2}{4m_N^2} \right) \approx 16m_N B , \quad (37)$$

which can give a very long maximal range of the distribution in configuration space, just as expected from the Schrödinger wave function.

There are other anomalous thresholds associated with the  $N\bar{N}$  branch point of  $W(s)$ . For instance, there are those due to the probability distribution of the proton in a deuteron, considered as a composite system of two

nucleons and a limited number of pions. Their position can easily be calculated exactly. For small values of  $B$ , we get  $g_1(m_D^2) \approx 16m_N(B + m_\pi)$ , if one pion is added. Anomalous thresholds can also come out of higher absorptive branch points in the  $s$ -channel of the form factor  $W$ .

Finally, we remark that the above considerations can be generalized to other amplitudes. Essentially only kinematics, crossing, and some analytic properties are needed. In all cases the anomalous thresholds are related to ordinary, absorptive thresholds in other amplitudes, which appear in the continuation into secondary Riemann sheets [21].

As we have pointed out before, due to the fact that anomalous thresholds are indirectly related to absorptive thresholds, there are *no* such singularities which are associated with the quark-gluon structure of hadrons, since there are no absorptive thresholds related to this structure. However, for hadrons which may be considered as loosely bound systems of heavy quarks, we can get a large mean square radius on the basis of appropriate weight functions along *hadronic* cuts [10, 31], even though they may be much higher in mass, and also as a consequence of possible *hadronic* anomalous thresholds.

## VI. COLORED AMPLITUDES

Having discussed only hadronic amplitudes describing observable consequences of the theory, we would like to add here some remarks about the analytic structure and the singularities of general Green's functions with colored channels. In particular, we will show that these colored amplitudes must have singularities at finite points, which can be associated with confined states in  $\mathcal{V}$  like quarks and gluons [22, 23]. Even though quarks and

transverse gluons are confined, we can have asymptotic states associated with these excitations, as well as corresponding poles in colored Green's functions. In our formulation of confinement, all colored states form quartet representations of BRST-algebra, and hence are not elements of physical space  $\mathcal{H}$ , which contains only singlets.

As an example for colored amplitudes, we consider the gluon propagator, which has been studied extensively. The structure function is defined as a Fourier Transform by

$$\begin{aligned} \int dx e^{ikx} \langle 0 | T A_a^{\mu\nu}(x) A_b^{\rho\sigma}(0) | 0 \rangle &= -i\delta_{ab} D(k^2 + i0) \\ &\times (k^\mu k^\rho g^{\nu\sigma} - k^\mu k^\sigma g^{\nu\rho} + k^\nu k^\sigma g^{\mu\rho} - k^\nu k^\rho g^{\mu\sigma}) \end{aligned} \quad (38)$$

with  $A^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$ . As before, we consider the linear, covariant gauges defined in Eq.(1). In the state space  $\mathcal{V}$  with indefinite metric, we write the spectral condition in the form [32]

$$\int d^4a e^{-ip \cdot a} (\Psi, U(a)\Phi) = 0, \quad (39)$$

for values of  $p$  *outside* of  $\overline{W^+} = \{p : p^0 \geq 0, p^2 \geq 0; p \in R^4\}$ , and for all  $\Psi, \Phi \in \mathcal{V}$ .

Lorentz covariance and spectral condition are sufficient to show that  $D(k^2 + i0)$  is the boundary value of an analytic function  $D(k^2)$ , which is regular in the cut  $k^2$ -plane, with a cuts along the positive real axis only. It is then an essential question to obtain the asymptotic behavior for  $k^2 \rightarrow \infty$  in all directions of the complex  $k^2$ -plane. In view of the asymptotic freedom of the theory, the asymptotic terms can be obtained with the help of renormalization group methods. In using the renormalization group, an assumption is made, which we have not used so far. We require that the general amplitude connects with the perturbative expression for  $g^2 \rightarrow +0$ , where  $g$  is the gauge

coupling parameter. The connection is needed only for the leading term. With this assumption, we find for  $k^2 \rightarrow \infty$  in all directions [23]:

$$-k^2 D(k^2, \kappa^2, g, \alpha) \simeq \frac{\alpha}{\alpha_0} + C_R(g^2, \alpha) \left( -\beta_0 \ln \frac{k^2}{\kappa^2} \right)^{-\gamma_{00}/\beta_0} + \dots . \quad (40)$$

The corresponding asymptotic terms for the discontinuity along the positive, real  $k^2$ -axis are then given by

$$-k^2 \rho(k^2, \kappa^2, g, \alpha) \simeq \frac{\gamma_{00}}{\beta_0} C_R(g^2, \alpha) \left( -\beta_0 \ln \frac{k^2}{|\kappa^2|} \right)^{-\gamma_{00}/\beta_0 - 1} + \dots . \quad (41)$$

In these relations, we have used the following definitions: The anomalous dimension of the gauge field is given by

$$\gamma(g^2, \alpha) = (\gamma_{00} + \alpha \gamma_{01}) g^2 + \dots \quad (42)$$

for  $g^2 \rightarrow +0$ , and for the renormalization group function we write, in the same limit,

$$\beta(g^2) = \beta_0 g^4 + \beta_1 g^6 + \dots . \quad (43)$$

Furthermore, we use the notation

$$\alpha_0 = -\gamma_{00}/\gamma_{01} . \quad (44)$$

For QCD, we have

$$\frac{\gamma_{00}}{\beta_0} = \frac{\frac{13}{2} - \frac{2}{3} N_F}{11 - \frac{2}{3} N_F} , \quad \gamma_{01} = (16\pi^2)^{-1} \frac{3}{4} , \quad (45)$$

where  $N_F$  is the number of quark flavours. We assume  $\beta_0 < 0$  corresponding to asymptotic freedom. Consequently, the exponent  $\gamma_{00}/\beta_0$  in Eqs.(40) and (41) varies from 13/22 for  $N_F = 0$  to 1/10 for  $N_F = 9$ , and from  $-1/16$  for

$N_F = 10$  to  $-15/2$  for  $N_F = 16$ . We have  $0 < \gamma_{00}/\beta_0 < 1$  for  $N_F \leq 9$  and  $\gamma_{00}/\beta_0 < 0$  for  $10 \leq N_F \leq 16$ ; for  $\gamma_{00}/\beta_0 = 1$ , our relations (40) and (41) would require modifications.

The parameter  $\kappa^2 < 0$  is a normalization point. We generally chose to normalize  $D$  so that

$$-k^2 D(k^2, \kappa^2, g, \alpha) = 1 \quad \text{for} \quad k^2 = \kappa^2. \quad (46)$$

With this normalization, the coefficient  $C_R(g^2, \alpha)$  for  $\alpha = 0$  is given by [22, 23]

$$\begin{aligned} C_R(g^2, 0) &= (g^2)^{-\gamma_{00}/\beta_0} \exp \int_{g^2}^0 dx \tau_0(x), \\ \tau_0(x) &\equiv \frac{\gamma(x, 0)}{\beta(x)} - \frac{\gamma_{00}}{\beta_0 x}, \end{aligned} \quad (47)$$

and hence  $C_R(g^2, 0) > 0$ . Certainly  $C_R(g^2, \alpha)$  is not identically zero. If there should be zero surfaces, a term proportional to  $(-\beta_0 \ln \frac{k^2}{\kappa^2})^{-1}$  becomes relevant in Eq.(41).

The remarkable property of the asymptotic terms in Eqs. (40) and (41) is their gauge independence except for the coefficients. Furthermore, their functional form is determined by one loop expressions.

From the asymptotic limit (40), we see that  $D(k^2)$  vanishes for  $k^2 \rightarrow \infty$  in all directions of the complex  $k^2$ -plane. Hence it cannot be a nontrivial entire function, at least for  $0 < g < g_\infty$ , where  $g_\infty$  is a possible first non-integrable singularity of  $\beta^{-1}(g^2)$ . There must be singularities on the positive real  $k^2$ -axis, and it is natural that these are associated with confirmed, unphysical states. Similar arguments can be given for the structure functions of the quark propagator.



We can write an unsubtracted dispersion representation for  $D(k^2)$  :

$$D(k^2, \kappa^2, g, \alpha) = \int_{-0}^{\infty} dk'^2 \frac{\rho(k'^2, \kappa^2, g, \alpha)}{k'^2 - k^2}, \quad (48)$$

and even a dipole representation exists

$$\begin{aligned} D(k^2, \kappa^2, g, \alpha) &= \int_{-0}^{\infty} dk'^2 \frac{\sigma(k'^2, \kappa^2, g, \alpha)}{(k'^2 - k^2)^2}, \\ \sigma(k^2, \kappa^2, g, \alpha) &= \int_{-0}^{k^2} dk'^2 \rho(k'^2, \kappa^2, g, \alpha). \end{aligned} \quad (49)$$

For  $\alpha = 0$ , the dipole representation has been used in order to give arguments for an approximately linear quark-antiquark potential under the condition  $\gamma_{00}/\beta_0 > 0$ , where  $\sigma(\infty) = 0$ , and  $\sigma(k^2) > 0$ ,  $\sigma'(k^2) = \rho(k^2) < 0$  for sufficiently large values of  $k^2$  [33, 34].

Under the restriction  $\gamma_{00}/\beta_0 > 0$  ( $N_F \leq 9$  for QCD), we find that  $D(k^2) - \frac{\alpha}{\alpha_0}$  vanishes faster than  $k^{-2}$  at infinity, so that we have the important sum rule [23] :

$$\int_{-0}^{\infty} dk^2 \rho(k^2, \kappa^2, g, \alpha) = \frac{\alpha}{\alpha_0}. \quad (50)$$

For  $\alpha = 0$ ,  $\gamma_{00}/\beta_0 > 0$ , we have a superconvergence relation [22]. It gives a rather direct connection between short and long distance properties of the theory, and has been used in order to give arguments for gluon confinement [16, 17].

## APPENDIX: REMARKS ABOUT PROOFS

We have seen that we can construct local hadronic fields as BRST-invariant operators in  $\mathcal{V}$ , and write Fourier representation of hadronic am-

plituds in terms of matrix elements of products of these fields. With BRST-methods, we define an invariant physical state space  $\mathcal{H}$  which, as a consequence of confinement, contains only hadrons. With the spectral conditions also referring to hadrons only, we have the input required in order to use the old methods for the derivation of dispersion representations as formulated in hadronic field theory. For completeness, we give in this appendix a very brief sketch of the essential ideas of these proofs, which are often hidden behind technical details.

The *Gap Method* [6] is applicable in cases where there is no unphysical region. Examples are  $\pi\pi$ -,  $\pi N$ - forward and near-forward scattering, some form factors like  $\pi\pi\gamma$ ,  $\pi NN$  in the  $N$ -channel, etc. [35, 36, 1]. As an example, let us consider  $\pi^0\pi^0$ - forward scattering. We can write the amplitude as

$$F(\omega) = \int_0^\infty dr F(\omega, r) , \quad (51)$$

with

$$\begin{aligned} F(\omega, r) &= 4\pi \frac{r \sin(\sqrt{\omega^2 - \mu^2})}{\sqrt{\omega^2 - \mu^2}} \\ &\times \int_0^\infty dx^0 e^{i\omega x^0} \langle p | [j(\frac{x}{2}), j(-\frac{x}{2})] | p \rangle , \end{aligned} \quad (52)$$

and  $j = (\square + \mu^2)\phi$ . For fixed  $r$ ,  $F(\omega, r)$  is analytic in the upper half  $\omega$ -plane, and  $ImF(\omega + i0, r) = 0$  for  $|\omega| < \mu$  due to the spectral conditions. Ignoring subtraction, we can write a Hilbert representation

$$F(\omega, r) = \frac{2\omega}{\pi} \int_\mu^\infty d\omega' \frac{ImF(\omega' + i0, r)}{\omega'^2 - \omega^2} . \quad (53)$$

For real  $|\omega| > \mu$ , we can perform the  $r$ -integration (51) on both sides, and get the corresponding dispersion relation for  $F(\omega)$ . Although some refinements are required, the method shows in a very simple way how local commutativity

and spectral conditions lead to a dispersion representation. Pole terms, like in  $\pi N$ - scattering, can also be handled by this method [5, 7, 1].

The *General Method* is required in the presence of unphysical regions, like  $NN$ -scattering [37] (even for  $t=0$ ),  $N\bar{N}\gamma$ -form factors in the  $N\bar{N}$ -channel, for fixed  $t$  amplitudes [38], to obtain  $t$ -analyticity (Lehmann ellipses) [8], and for  $st$ -analyticity [39]. There are many technical details involved in the derivation of dispersion representations, like continuations in mass variables, for example, but the main problem is to construct the largest region of holomorphy obtainable on the basis of retarded and advanced functions like

$$F_{\pm}(K) = \pm \frac{i}{(2\pi)^3} \int d^4x e^{-iK \cdot x} \theta(\pm x^0) \langle p' | \left[ j^{\dagger}\left(\frac{x}{2}\right), j\left(-\frac{x}{2}\right) \right] | p \rangle , \quad (54)$$

with  $K = \frac{1}{2}(k + k')$ ,  $k + p = k' + p'$  .

Due to local commutativity, the functions  $F_{\pm}(K)$  are analytic in the wedges

$$W^{\pm} = \{K : \text{Im}K^0 > 0 \text{ or } < 0, \quad (\text{Im}K)^2 > 0; \quad \text{Re}K \in R^4\} . \quad (55)$$

From the spectral conditions, we find that  $F_+(K) = F_-(K)$  for  $D \in R^4$ , where  $D$  is a real domain, and where this equality may be in the sense of distributions. As a special case of the *Edge of the Wedge Theorem* [1], we can then show that there exists an analytic function  $F(K)$ , which coincides with  $F_{\pm}(K)$  in the wedges  $W^{\pm}$  respectively, and which is holomorphic in the domain  $W \cup N(D)$ , with  $W = W^+ \cup W^-$ . Here  $N(D)$  is a finite, complex neighborhood of  $D$ . If we then construct the *Envelope of Holomorphy*  $E(W \cup N(D))$ , we get the largest possible region of analyticity given the assumptions made. In the original paper [1], a generalized semitube has been used, for which the envelope was known., This method gives boudary points of the

envelope in important cases. A complete construction of the envelope, using the continuity theorem, was given in [2].

Independently, in [5], elaborate distribution and analytic methods were used in order to get a subdomain of  $E$ , directly on the basis of  $W^\pm$  and  $D$ . For the special problem with one four-vector considered here, one can use methods from the theory of distributions and differential equations in order to give a representation of functions which are holomorphic in  $E$  [4].

The limitations of the proofs for dispersion representation are due to the lack of input from unitarity, and often can be related to conditions for the absence of unphysical anomalous thresholds. Some improvements are possible using aspects of two-particle unitarity, but in general multiparticle unitarity and analytic properties of multiparticle amplitudes are required for further enlargements of the domain of holomorphy.

For any fixed  $t < 0$ , and for arbitrary binary reactions, it can be shown that the amplitude is holomorphic outside of a large circle in the cut  $s$ -plane, so that one can always prove crossing relations [40].

As is evident from the preceding discussion, the interesting proposal of double dispersion relations [41] has not been proven. They are compatible with hadronic perturbation theory in lower orders. Although it may not be a valid approach in QCD, hadronic perturbation theory is a useful tool for locating certain singularities of physical amplitudes.

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## Figure Captions

**Fig. 1.** Vertex Function  $W(s)$ .

**Fig. 2.** Inelastic amplitude  $G(s, t)$  (a) and relevant pole term (b) in the  $t$ -channel.

**Fig. 3.** Branch points of  $W(s)$  and  $W^{II}(s)$ . The continuation is with respect to the  $N\overline{N}$ -threshold at  $s = 4m_N^2$ .

**Fig. 4.** Anomalous threshold of  $W(s)$  at  $s = g(x)$  for  $x > 2m_N^2$ . The branch line runs from  $g(x)$  to  $4m_N^2$  in sheet I (physical sheet), and then from  $4m_N^2$  to  $-\infty$  in sheet II (dotted line). The meson branch lines starting at  $s_\pi < 4m_N^2$  have not been drawn.